Discrete Mathematics 49 (1984) 315–317
North-Holland

NOTE

EQUIVALENCE BETWEEN THE MINIMUM COVERING PROBLEM AND THE MAXIMUM MATCHING PROBLEM

Ján PLESNÍK
Matematicko-fyzikálna fakulta UK, 842 15 Bratislava, Czechoslovakia

Received 19 October 1982
Revised 19 August 1983

The minimum covering problem in weighted graphs with n vertices is transformed in time $O(n^2)$ to the maximum matching problem with n or $n+1$ vertices, and conversely.

Introduction

Given a graph G (undirected, without loops and multiple edges), $V(G)$ and $E(G)$ denote its vertex set and edge set, respectively. A set $S \subseteq E(G)$ is called a covering of G if any vertex of G is incident to at least one edge of S; S is called a matching of G if no two edges of S are adjacent. If every edge $ij \in E(G)$ has prescribed a cost c_{ij} (a real number), then the cost of S is defined to be the sum of its element costs. The minimum covering problem is to find a c-minimum covering of G. The maximum matching problem is to find a c-maximum matching of G.

Both these problems are well known (see e.g. [1, 4, 5]). A polynomial algorithm for the maximum matching problem was developed by Edmonds (1965); for an $O(n^3)$-implementation see e.g. [4]. White (1967) (see [5]) used the Edmonds method and gave a polynomial algorithm for the minimum covering problem. In the special case when all the costs are equal the problems are equivalent as shown by Gallai [3] and Norman and Rabin [6]. Although Tutte (1954) and Edmonds (1967) have found transformations of certain degree-constrained subgraph problems to the maximum matching problem, obtained problems are often very large (cf. [4]) and therefore direct algorithms were developed [2].

In this note we shall show the equivalence of the minimum covering problem and the maximum matching problem in the sense that either problem in an n-vertex graph will be transformed, in time $O(n^2)$, to the other problem in a complete graph with n or $n+1$ vertices.

1. From the minimum covering problem to the maximum matching problem

One can suppose that G is a graph without isolated vertices and that all its costs are nonnegative. Our transformation consists of the following steps.

Step 1.1. For each vertex $i \in V(G)$ compute $d_i = \min\{c_{ij} \mid ij \in E(G)\}$ and choose an edge e_i incident to i and with $c(e_i) = d_i$.

Step 1.2. On the set $V(G)$ form a complete graph with costs

$$\hat{c}_{ij} = \begin{cases} \min\{c_{ij}, d_i + d_j\} & \text{if } ij \in E(G), \\ d_i + d_j & \text{else.} \end{cases}$$

Moreover, if $|V(G)|$ is odd, then add a new vertex w and for each $i \in V(G)$ an edge iw with cost $\hat{c}_{iw} = d_i$. The obtained complete graph will be denoted by $\hat{G}$.

Step 1.3. Choose a number $A > \max\{\hat{c}_{ij} \mid ij \in E(\hat{G})\}$ and for every $ij \in E(\hat{G})$ define $\bar{c}_{ij} = A - \hat{c}_{ij}$.

Step 1.4. Find a $\bar{c}$-maximum matching M of $\hat{G}$.

Step 1.5. Construct a set S by choosing one or two edges of G for every edge $ij \in M$ as follows:

(a) If $ij \in E(G)$ and $c_{ij} \leq d_i + d_j$ then choose the edge ij. (We see that $c_{ij} = \hat{c}_{ij}$.)

(b) If either $ij \in E(G)$ and $c_{ij} > d_i + d_j$, or $ij \notin E(G)$ and $i, j \in V(G)$, then choose the edges e_i and e_j (defined in Step 1.1). (Clearly, $c(e_i) + c(e_j) = d_i + d_j = \hat{c}_{ij}$.)

(c) If one of the vertices i, j does not belong to $V(G)$, say $j = w$, then choose the edge e_i. (We see that $c(e_i) = d_i = \hat{c}_{iw}$.)

Proposition 1. *The set S of edges chosen in Step* 1.5 *is a c-minimum covering of G and $c(S) = \hat{c}(M) = A\hat{n}/2 - \bar{c}(M)$, where $\hat{n} = |V(\hat{G})|$.*

Outline of proof. As each $\bar{c}_{ij} > 0$, M is a $\hat{c}$-minimum perfect matching of $\hat{G}$ and hence S is a covering of G with $c(S) \leq \hat{c}(M)$. On the other hand, given a c-minimum covering S' of G, one can construct a perfect matching M' of $\hat{G}$ with $\hat{c}(M') = c(S')$. Namely, one can suppose that the subgraph of G formed by S' consists of stars (trees of radius one). Constructing M', we at first choose one edge from every star and then the remaining vertices are paired arbitrarily. Since $\hat{c}(M) \leq \hat{c}(M')$, the proof follows. □

2. From the maximum matching problem to the minimum covering problem

One can suppose that all costs of G are nonnegative. Then we shall proceed as follows.

Step 2.1. By adding new edges form a complete graph $\hat{G}$ with $V(\hat{G}) = V(G)$ if $|V(G)|$ is even and with $V(\hat{G}) = V(G) \cup \{w\}$ if $|V(G)|$ is odd, where w is a new vertex. Define costs

$$\hat{c}_{ij} = \begin{cases} c_{ij} & \text{if } ij \in E(G), \\ 0 & \text{else.} \end{cases}$$

Step 2.2. Letting $\hat{c}_{\max} = \max\{\hat{c}_{ij} \mid ij \in E(\hat{G})\}$, choose a number $B > 2\hat{c}_{\max}$. Then define $\bar{c}_{ij} = B - \hat{c}_{ij}$ for all $ij \in E(\hat{G})$.

Step 2.3. Find a $\bar{c}$-minimum covering Y of $\hat{G}$ and construct a set $M = Y \cap E(G)$.

Proposition 2. *The set M obtained in Step* 2.3 *is a c-maximum matching of G and* $c(M) = \hat{c}(Y) = B\hat{n}/2 - \bar{c}(Y)$, *where* $\hat{n} = |V(\hat{G})|$.

Outline of proof. As each $\bar{c}_{ij} > 0$, every component of the subgraph of $\hat{G}$ formed by Y is a star. Moreover, as B is sufficiently large and $\hat{n}$ is even, every one of these stars has only two vertices (otherwise a better covering than Y can be made from Y by deleting two edges and adding one edge). Thus Y is a $\bar{c}$-minimum perfect matching of $\hat{G}$ and therefore it is a $\hat{c}$-maximum perfect matching of $\hat{G}$. Hence $Y \cap E(G)$ is a c-maximum matching of G. □

References

[1] N. Christofides, Graph Theory: An Algorithmic Approach (Academic Press, London, 1975).

[2] J. Edmonds, Some well-solved problems in combinatorial optimization, in: Combinatorial Programming: Methods and Applications (Reidel, Dordrecht, 1975) 285–301.

[3] T. Gallai, Über extreme Punkt- und Kantenmengen, Ann. Univ. Sci. Budapest. Eötvös Sect. Math. 2 (1959) 133–138.

[4] E.L. Lawler, Combinatorial Optimization: Networks and Matroids (Holt, Rinehart and Winston, New York, 1976).

[5] E. Minieka, Optimization Algorithms for Networks and Graphs (Marcel Dekker, New York, 1978).

[6] R.Z. Norman and M. Rabin, Algorithm for a minimal cover of a graph, Proc. Amer. Math. Soc. 10 (1959) 315–319.